On the Igusa modular form of weight 10 (Automorphic forms, trace formulas and zeta functions)

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On the Igusa modular form of weight 10

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Abstract. This paper is related to the authors' talk at the RIMS conference 2011 on: *Automorphic forms, trace formulas and zeta functions* in Kyoto. The Igusa modular form of weight 10 is the unique Siegel modular form which is a Borcherds and a Saito-Kurokawa lift.

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1 Introduction

The Igusa modular form χ_{10} appeared first in the famous theorem of Jun-ichi Igusa about the generators of graded algebra of Siegel modular forms of even weight and degree 2 (see [Ig1]). The algebra is equal to

$$\mathbb{C}[E_4^2, E_6^2, \chi_{10}, E_{12}^2]. \tag{1.1}$$

We normalized the Siegel type Eisenstein series E_k^2 of weight k such that the Fourier coefficient related to 0-dim cusp at infinity is one. The Igusa modular form χ_{10} is a cusp form of weight 10. Igusa introduced the form in terms of Eisenstein series ([Ig1], page 192).

$$\chi_{10} := -43867 \cdot 2^{-12} \cdot 3^{-5} \cdot 5^{-2} \cdot 7^{-1} \cdot 53^{-1}(E_4^2 E_6^2 - E_{10}^2).$$

It is known that χ_{10} is a Saito-Kurokawa lift ([Za]) and a Borcherds lift ([GN1], [GN2]).

The square root of this modular form is related to the denominator formula for a generalized Borcherds-Kac-Moody super algebra (Gritsenko, Nikulin). Moreover it is as a partition function of BPS dyons in the toroidally compactified heterotic string theory. To study a generalized Kac-Moody algebra one has to know the imaginary simple roots and the multiplicities of all positive roots. It is absolutely crucial that the underlying modular form has a degenerate Fourier expansion (Saito-Kurokawa lift) and an infinite product (Borcherds lift). We refer to ([CD], [CV]) for more details. The following theorem states that there are no other Siegel modular forms of degree 2 which are Borcherds and Saito-Kurokawa lifts .

Theorem *Let F be a Siegel modular form of degree 2. If F is a Borcherds lift and a Saito-Kurokawa lift, then F is proportional to the Igusa modular form.*

We note that the Borcherds lift is multiplicative and the Saito-Kurokawa lift additive.

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2 Siegel modular forms, Witt operator and Taylor expansions

For an introduction to the theory of Siegel modular forms we refer to Klingen's book ([Kl]). Let Γ_n be the Siegel modular group and $\mathfrak{H}_n$ the upper half space of degree n:

$$\begin{aligned}\Gamma_n &:= \left\{\gamma \in \mathrm{GL}_{2n}(\mathbb{Z}) \mid {}^t\gamma \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix} \gamma = \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix}\right\} \\ \mathfrak{H}_n &:= \{Z \in \mathrm{M}_n(\mathbb{C}) \mid {}^tZ = Z, \mathrm{Im}(Z) > 0\},\end{aligned}$$

where $\mathbf{0}_n$ (respectively $\mathbf{1}_n$) is the zero (respectively identity) matrix of degree n. Then we denote by $M_k(\Gamma_n)$ the space of Siegel modular forms of weight k on Γ_n and by $S_k(\Gamma_n)$ the subspace of cusp forms. In the case $n = 1$ we usually drop the index and for $n = 2$ which we are mainly interested in we often write (τ_1, z, τ_2) for a point

$$\begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix} \in \mathfrak{H}_2.$$

Next we introduce two useful operators. Let $F \in M_k(\Gamma_2)$. Define

$$\begin{aligned}\Phi(F)(\tau) &:= \lim_{y\to\infty} F(\tau, 0, iy) \qquad (\tau \in \mathfrak{H}_1), \\ \mathcal{W}(F)(\tau_1, \tau_2) &:= F(\tau_1, 0, \tau_2) \qquad (\tau_1, \tau_2 \in \mathfrak{H}_1).\end{aligned}$$

Then $\Phi(F) \in M_k(\Gamma)$ and $\mathcal{W}(F) \in \mathrm{Sym}^2(M_k(\Gamma))$. The operator Φ (respectively $\mathcal{W}$) is called the *Siegel* (respectively *Witt*) operator. Then $S_k(\Gamma_2) = \{F \in M_k(\Gamma_2) \mid \Phi(F) = 0\}$.
Let $f_1, f_2, \ldots, f_d$ be a basis of newforms of S_k and $f_0 = e_k$. Here e_k denotes the elliptic Eisenstein series with constant term $a(0) = 1$.

Then we define

$$\mathrm{Sym}^2(M_k(\Gamma))^D := \left\{\sum_{i=0}^{d} \alpha_i\, f_i \otimes f_i \mid \alpha_i \in \mathbb{C}\right\}. \tag{2.1}$$

By $\mathrm{Sym}^2(S_k(\Gamma))^D$ we denote the cuspidal part.

A Siegel modular form $F \in M_k(\Gamma_2)$ admits the Fourier expansion

$$F(\tau_1, z, \tau_2) = \sum_{n,r,m\in\mathbb{Z}} A_F(n, r, m)\; \mathbf{e}(n\tau_1 + rz + m\tau_2),$$

where we put $\mathbf{e}(z) = \exp(2\pi i z)$ for $z \in \mathbb{C}$. Note that $A_F(n, r, m) = 0$ unless $n, m, 4nm - r^2 \geq 0$. We also use the following shortcuts: $q := \mathbf{e}(\tau)$, $q_1 := \mathbf{e}(\tau_1)$, $\zeta := \mathbf{e}(z)$, $q_2 := \mathbf{e}(\tau_2)$. It is easy to see that:

$$\Phi(F)(\tau) = \sum_{n=0}^{\infty} A_F(n, 0, 0)\, q^n \tag{2.2}$$

$$\mathcal{W}(F)(\tau_1, \tau_2) = \sum_{n,m=0}^{\infty} \left(\sum_r A_F(n, r, m)\right) q_1^n q_2^m. \tag{2.3}$$

We define the order of the q-expansion of a modular form $F \in M_k(\Gamma_2)$ by

$$ord(F) := min \{n \in \mathbb{N}_0 \mid A_F(n,r,m) \neq 0\}.$$

Remark 2.1. If $ord(F) \geq 2$, then $F \notin \mathrm{Sym}^2(M_k(\Gamma))^D$.

Let k be even. Then $F \in M_k(\Gamma)$ has the Taylor expansion

$$F(\tau_1, z, \tau_2) = \sum_{l=0}^{\infty} \Psi_{2l}(\tau_1, \tau_2)\, z^{2l}. \tag{2.4}$$

It is clear that Ψ_0 is the image of the Witt operator and an element of $\mathrm{Sym}^2(M_k(\Gamma))$. Moreover if Ψ_0 is identically zero then $\psi_2 \in \mathrm{Sym}^2(S_{k+2}(\Gamma))$.

Finally let E_k^n denote the Siegel-type Eisenstein series on Γ_n, normalized by $\Phi^n(E_k^n) = 1$. Here Φ^n denotes the n-th iteration of the Φ operator. Let $E_k^n(f)$ denote the Klingen Eisenstein series attached to $f \in S_k(\Gamma), f \neq 0$. Note that $\Phi^{n-1}(E_k^n(f)) = f$. Let further $M_k^{2,0}$ be the 1-dim space generated by Siegel Eisenstein series of weight k and degree 2, let $M_k^{2,1}$ be the space generated by all Klingen type Eisenstein series of weight k and degree 2 and let $M_k^{2,2} = S_k(\Gamma_2)$. Then

$$M_k(\Gamma_2) = M_k^{2,0} \oplus M_k^{2,1} \oplus M_k^{2,2}. \tag{2.5}$$

The direct sum is related to the Petersson scalar product. Moreover this decomposition is respected by the Siegel Φ operator. Let $F \in M_k(\Gamma_2)$ with decomposition $F_0 + F_1 + F_2$. Then

$$\begin{aligned} \Phi(F) &= \Phi(F_0) + \Phi(F_1) + \Phi(F_2) && (2.6)\\ &= c_1\, E_k + c_2\, f \quad (c_1, c_2 \in \mathbb{C},\ f \in S_k(\Gamma)). && (2.7) \end{aligned}$$

3 Saito-Kurokawa lifts

One can find an overview in Zagier's Bourbaki article [Za]. Let k be an even integer. Then there exists an injective linear map

$$SKL : M_{2k-2}(\Gamma) \longrightarrow M_k(\Gamma_2), \tag{3.1}$$

where Hecke eigenforms f map to Hecke eigenforms $F = SKL(f)$. For a Hecke eigenform f, the spinor L-function $Z(SKL(f), s)$ is given by

$$Z(SKL(f), s) = \zeta(s-k+1)\, \zeta(s-k+2)\, L(f, s),$$

where $L(f, s)$ is the Hecke L-function of f and $\zeta(s)$ denotes the Riemann zeta function. We are interested in the image of the lifting, which is given by the so-called Maass Spezialschar:

$$M_k^{Spez} := \left\{ F \in M_k(\Gamma_2) \;\middle|\; A_F(n,r,m) = \sum_{d \in \mathbb{N},\, d|(n,r,m)} d^{k-1}\, A_F\left(\frac{nm}{d^2}, \frac{r}{d}, 1\right) \right\}. \tag{3.2}$$

Here (n,r,m) denotes the greatest common divisor of n,r,m (We put 1:=(0,0,0)). To prove our main result we use the following properties of the Maass Spezialschar. If $F \in M_k^{Spez}$, then F is non-trivial iff Ψ_0^F or Ψ_2^F is not identically zero. Moreover

$$\Psi_0^F \in \mathrm{Sym}^2(M_k(\Gamma))^D. \tag{3.3}$$

If Ψ_0^F is identically zero then

$$\Psi_2^F \in \mathrm{Sym}^2(S_{k+2}(\Gamma))^D. \tag{3.4}$$

Remark 3.1. Let $F \in M_k(\Gamma_2)$ has the decomposition $F_0 + F_1 + F_2$ as described before. If F_1 is non-trivial, then F is not in the Spezialschar.

4 Borcherds lifts

Roughly speaking a Borcherds lift BL is a correspondence between modular forms of weight $1 - \frac{m}{2}$ on $\mathfrak{H}$ with possible singularities at the cusps and certain meromorphic automorphic forms with possible character on symmetric domains of type IV related to orthogonal groups $O(2,m)$ ($m \in \mathrm{N}$) ([Bo1],[Bo2], [Bo3]). We note that

$$BL(f+g) = BL(f) \cdot BL(g).$$

Lifts to Siegel modular forms of degree 2 are related to the case $m = 3$, where the image is uniquely (up to a scalar) determined by the divisor

$$div\,(BL(f)) = \sum_{d \in \mathcal{D}} n_d\, H_d. \tag{4.1}$$

Here $\mathcal{D}$ is the set of all positive integers congruent to 0 or 1. The sum is finite and $n_d \in \mathbb{Z}$. The H_d are the Humbert surfaces (see also the following subsection), for general m they are called Heegner divisor. The image could be an element of $M_k(\Gamma_2, v)$, a Siegel modular form with the unique non-trivial character v on Γ_2.

Remark 4.1. The coefficients of the principal part of the input function are related to the n_d. A priori it is not clear when the nontrivial character in the image occurs. Moreover even when not all coefficients in the principal part are non-negative, the image could be holomorphic.

4.1 Humbert surfaces

Let

$$Q := \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & -2 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}.$$

Put $Q(X,Y) := {}^tXQY$ and $Q[X] := Q(X,X)$ for $X,Y \in \mathbb{C}^5$. For $Z = (\tau_1, z, \tau_2) \in \mathfrak{H}_2$ put $\widetilde{Z} := {}^t(-\tau_1\tau_2 + z^2, \tau_1, z, \tau_2, 1) \in \mathbb{C}^5$. Note that $Q[\widetilde{Z}] = 0$ and $Q(\widetilde{Z}, \overline{\widetilde{Z}}) = 4\det(\mathrm{Im}(Z)) > 0$. There exists a homomorphism $\iota\colon \mathrm{Sp}_2(\mathbb{R}) \to O(Q)_{\mathbb{R}}$ such that $\widetilde{g\langle Z\rangle} = j(g,Z)^{-1}\iota(g)\widetilde{Z}$ for $g \in \mathrm{Sp}_2(\mathbb{R})$ and $Z \in \mathfrak{H}_2$.

Let $L := \mathbb{Z}^5, L^* := Q^{-1}L$ and $L^*_{\text{prim}} := \{\lambda \in L^* \mid n^{-1}\lambda \notin L^* \text{ for any integer } n > 1\}$. For an integer $d \in \mathbb{Z}$, let

$$\mathcal{H}_d := \sum_{X \in \mathcal{L}_d} \left\{ Z \in \mathfrak{H}_2 \mid Q(X, \widetilde{Z}) = 0 \right\},$$

where $\mathcal{L}_d := \{X \in L^*_{\text{prim}} \mid Q[X] = -d/2\}$. Note that $\mathcal{H}_d = 0$ unless $d > 0$ and $d \equiv 0$ or $1 \pmod 4$. Since L^*_d is $\iota(\Gamma_2)$-invariant, $\mathcal{H}_d$ is Γ_2-invariant. Denote by H_d the image of $\mathcal{H}_d$ in $\Gamma_2\backslash\mathfrak{H}_2$ by the natural projection $\mathfrak{H}_2 \to \Gamma_2\backslash\mathfrak{H}_2$. The divisor H_d of $\Gamma_2\backslash\mathfrak{H}_2$ is called the *Humbert surface* of discriminant d. It is known that H_d is nonzero and irreducible if $d \equiv 0$ or $1 \pmod 4$ (see [Ge2], page 212, Theorem 2.4; see also [GH], Section 3). Note that

$$\mathcal{H}_1 = \bigcup_{\gamma \in \Gamma_2} \gamma\{(\tau_1, 0, \tau_2) \mid \tau_1, \tau_2 \in \mathfrak{H}\}$$

$$\mathcal{H}_4 = \bigcup_{\gamma \in \Gamma_2} \gamma\{(\tau, z, \tau) \mid \tau \in \mathfrak{H}, z \in \mathbb{C}\}.$$

4.2 Properties of Borcherds lifts and examples

Recently [HM] we found an explicit description of the Borcherds lifts related to single Heegner divisors. As a by-product one can see that the character is only related to the divisors H_1 and H_4.

Theorem 4.2.

(i) For each positive integer d with $d \equiv 0$ or $1 \pmod 4$, there exists an $F_d \in M_{k_d}(\Gamma_2, v^{\alpha_d})$ with $\alpha_d \in \{0,1\}$ satisfying $\mathrm{div}(F_d) = H_d$.

(ii) We have $F_1 \in S_5(\Gamma_2, v), F_4 \in S_{30}(\Gamma_2, v)$ and $F_d \in M_{k_d}(\Gamma_2)$ if $d > 4$.

(iii) A Borcherds lift $F \in M_k(\Gamma_2, v^\alpha)$ $(\alpha \in \{0,1\})$ is a constant multiple of $\prod_d F_d^{A(d)}$, where d runs over the positive integers with $d \equiv 0$ or $1 \pmod 4$, and $A(d)$ is a nonnegative integer ($A(d) = 0$ except for a finite number of d) satisfying $A(1) + A(4) \equiv \alpha \pmod 2$.

Here $S_k(\Gamma_2), v)$ denotes the cuspidal subspace of $M_k(\Gamma_2), v)$

It is well-known that $\dim S_{10}(\Gamma_2) = 1$ (see also [Kl]). Hence χ_{10} is proportional to F_1^2.

Remark 4.3. The Borcherds lifts in $M_k(\Gamma_2)$ with $k \leq 60$ are listed as follows:

Borcherds lift	weight	divisor
F_1^{2a} $(1 \le a \le 6)$	$10a$	$2aH_1$
$F_1^{2a+1}F_4$ $(1 \le a \le 2)$	$10a + 35$	$(2a+1)H_1 + H_4$
$F_1^{2a}F_5$ $(1 \le a \le 3)$	$10a + 24$	$2aH_1 + H_5$
F_4^2	60	$2H_4$
F_5^2	48	$2H_5$
F_8	60	H_8

The table shows that every Borcherds lift of weight less than or equal to 60 is a monomial of F_1, F_4, F_5 and F_8. We also see that there is no holomorphic Borcherds lift of weight 12.

Assume that $F \in M_k(\Gamma_2)$ is a Borcherds lift. Then $\Phi(F)$ is proportional to a power Δ^r of the modular discriminant Δ with $r \ge 0$.

5 Proof of the Theorem

In the following we give a sketch of the proof of the main theorem. The complete proof will appear elsewhere. Let $F \in M_k(\Gamma_2), F \neq 0$. Let F be a Borcherds lift (BL) and Saito-Kurokawa lift (SKL). First of all we can assume that the weight is even (SKL). This implies that $k \ge 4$. The structure theorem (BL) leads to

(5.1)
$$F \sim \prod_{d \in \mathcal{D}} F_d^{n_d}.$$

The product is finite, $n_1 + n_4 \equiv 0 \pmod 2$ and $n_d \in \mathbb{N}_0$. The symbol $\sim$ indicates that two function are equal up to a non-zero constant.

Remark 5.1. A refined analysis of the modular forms F_d shows that

$$ord(F_1) = \frac{1}{2}, ord(F_4) = \frac{3}{2}.$$

If $d \ge 5$ then $ord(F_d) \ge 2$ iff d is a square and $ord(F_d) = 0$ otherwise.

Since F is also a SKL we have $ord(F) \le 1$. This leads to

(5.2)
$$F \sim F_1^{\alpha} \cdot \prod_{d \ge 5,\ d \text{ not a square}} F_d^{n_d} \quad (\alpha = 0, 2).$$

Put $G := F/F_1^{\alpha}$. Since G is a BL and not a cusp form we have [HM]

$$\Phi(G) \sim \Delta^r \quad (r = \frac{k - 5\alpha}{12} \in \mathbb{N}).$$

Then it is easy to see that

$$\mathcal{W}(G) \sim \Delta^r \otimes E_k + E_k \otimes \Delta^r + \text{ cuspidal}.$$

This shows that, if $\alpha = 0$, then $G = F$ is not a SKL, a contradiction. Thus we have $\alpha = 2$. Finally the case $\alpha = 2$ remains. We show that $r \geq 1$ is not possible (then the theorem is proven).

Let in the following $F \sim F_1^2 \dot{G}$, with $\Phi(G) = \Delta^r$ $(r \geq 1)$. Then Ψ_0^F is identically 0. Since F is a SKL and not identically zero, we can assume that $\Psi_2^F \neq 0$ and that

$$\Psi_2^F \in \mathrm{Sym}^2(S_{k+2}(\Gamma))^D. \tag{5.3}$$

Since the second Taylor coefficient of F_1^2 is proportional to $\Delta \otimes \Delta$ we obtain

$$\Psi_2^F \sim (\Delta \otimes \Delta) \cdot \mathcal{W}(G). \tag{5.4}$$

On the other hand $\mathcal{W}(G)$ can be expressed in terms of the modular function j and the primitive modular polynomial. This can be directly proven by comparing the weights and the divisors on $\mathfrak{H} \times \mathfrak{H}$.

For $m \in \mathbb{Z}_{>0}$, let $\mathcal{M}_m^*$ be the set of primitive matrices in $\mathrm{M}_2(\mathbb{Z})$ of determinant m. As is well-known, there exists a polynomial Φ_m^* in $\mathbb{Z}[X, Y]$, called the primitive modular polynomial of degree m, such that

$$\prod_{M \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{M}_m^*} (X - j(M\langle\tau\rangle)) = \Phi_m^*(X, j(\tau)).$$

Here $\tau \mapsto M\langle\tau\rangle$ denotes the action on $\mathfrak{H}$. The degree of $\Phi_m^*(X, Y)$ in X is larger than m for $m > 1$. Then

$$\mathcal{W}(G)(\tau_1, \tau_2) \sim (\Delta^r(\tau_1) \otimes \Delta^r(\tau_2)) \prod_{n>0} \Phi_n^*(j(\tau_1), j(\tau_2))^{a(n)}, \tag{5.5}$$

where $a(n) \in \mathrm{N}_0$. Hence we obtain

$$\Psi_2^F(\tau_1, \tau_2) \sim (\Delta^{r+1}(\tau_1) \otimes \Delta^{r+1}(\tau_2)) \prod_{n>0} \Phi_n^*(j(\tau_1), j(\tau_2))^{a(n)}. \tag{5.6}$$

Combining this property with (5.3) leads to a contradiction by employing well-known properties of the modular polynomial, multiplicative properties of the Fourier coefficients of primitive Hecke eigenforms and the explicit Fourier expansion of the Δ-function.

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